

# BOREL CARDINAL INVARIANT PROPERTIES OF COUNTABLE BOREL EQUIVALENCE RELATIONS

SAMUEL COSKEY AND SCOTT SCHNEIDER

**ABSTRACT.** Boykin and Jackson recently introduced a property of countable Borel equivalence relations called Borel boundedness, which they showed is closely related to the unions problem for hyperfinite relations. In this paper, we introduce a family of properties of countable Borel equivalence relations which correspond to combinatorial cardinal invariants of the continuum in the same way that Borel boundedness corresponds to the bounding number  $\mathfrak{b}$ . We analyze some of the basic behavior of these properties, showing for instance that the property corresponding to the splitting number  $\mathfrak{s}$  coincides with smoothness. We then settle many of the implication relationships between the properties; these relationships turn out to be closely related to (but not the same as) the Borel Tukey ordering on cardinal invariants.

## 1. INTRODUCTION

The Borel equivalence relation  $E$  on the standard Borel space  $X$  is called *hyperfinite* if  $E$  can be written as a countable increasing union  $E = \bigcup_n F_n$  of finite Borel equivalence relations  $F_n$ . Dougherty, Jackson, and Kechris developed the basic theory of hyperfinite Borel equivalence relations in [DJK94], where they asked the following fundamental question that remains open:

1.1. *Question* ([DJK94]). Is the countable increasing union of hyperfinite Borel equivalence relations hyperfinite?

We refer to this as the *unions problem*, and call a countable increasing union of hyperfinite Borel equivalence relations *hyper-hyperfinite*. In [BJ07], Boykin and Jackson introduced the notion of *Borel boundedness* and showed that it is closely related to the unions problem.

1.2. **Definition** ([BJ07]). Let  $E$  be a countable Borel equivalence relation on the standard Borel space  $X$ . Then  $E$  is said to be *Borel bounded* if for every Borel function  $\phi: X \rightarrow \omega^\omega$ , there exists a Borel homomorphism  $\psi: X \rightarrow \omega^\omega$  from  $E$  to  $=^*$  such that  $\phi(x) \leq^* \psi(x)$  for all  $x \in X$ .

---

2000 *Mathematics Subject Classification.* 03E15; 03E17 .

*Key words and phrases.* Borel equivalence relations, cardinal characteristics of the continuum.

Here  $=^*$  and  $\leq^*$  are the relations of eventual equality and eventual boundedness on  $\omega^\omega$ . A family of functions  $\mathcal{F} \subset \omega^\omega$  is *unbounded* if there is no function  $\beta \in \omega^\omega$  such that  $\alpha \leq^* \beta$  for all  $\alpha \in \mathcal{F}$ . The *bounding number*,  $\mathfrak{b}$ , is defined to be the minimal cardinality of an unbounded family  $\mathcal{F} \subset \omega^\omega$ .

Of course, no countable family  $\mathcal{F} = \{\alpha_n \in \omega^\omega : n \in \omega\}$  can be unbounded, since the function  $\beta \in \omega^\omega$  defined by

$$(1.3) \quad \beta(n) := \max_{k \leq n} \alpha_k(n)$$

eventually dominates each  $\alpha_n \in \mathcal{F}$ . Hence  $\aleph_0 < \mathfrak{b}$ , and so for each Borel function  $\phi: X \rightarrow \omega^\omega$ , there trivially exists an  $E$ -invariant function  $\psi: X \rightarrow \omega^\omega$  such that for every  $x \in X$  and  $y \in [x]_E$ , we have  $\phi(y) \leq^* \psi(x)$ . But for  $E$  to be Borel bounded, the bounding functions  $\psi(x)$  for the countable families  $\phi([x]_E)$  cannot depend in an essential way on an enumeration of the class  $[x]_E$ .

Boykin and Jackson observed that every hyperfinite Borel equivalence relation is Borel bounded, and then established the following basic link between Borel boundedness and the unions problem.

**1.4. Theorem ([BJ07]).** *Let  $E$  be a countable Borel equivalence relation on the standard Borel space  $X$ . If  $E$  is hyper-hyperfinite and Borel bounded, then  $E$  is hyperfinite.*

What they left open, however, in addition to the unions problem itself, is the following important question.

**1.5. Question.** Is Borel boundedness equivalent to hyperfiniteness?

There is no known example of a non-hyperfinite countable Borel equivalence relation that is Borel bounded, and the only known examples of countable Borel equivalence relations that are *not* Borel bounded have been established by Thomas under the additional assumption of Martin's Conjecture on degree invariant Borel maps [Tho09].

After seeing Theorem 1.4, Thomas asked whether other cardinal invariants could be used in a role similar to that played by  $\mathfrak{b}$  in the definition of Borel boundedness. To explain, suppose that  $E$  is a countable Borel equivalence relation on the standard Borel space  $X$ . Many cardinal invariants of the continuum can be defined as the minimal cardinality of a subset of  $\omega^\omega$  (or of  $\mathcal{P}(\omega)$ , or  $[\omega]^\omega$ ) having some given combinatorial property  $P$ . Since each such cardinal is uncountable, it will trivially be the case that for every Borel function  $\phi: X \rightarrow \omega^\omega$ , there is an  $E$ -invariant function  $\psi: X \rightarrow \omega^\omega$  such that for each  $x \in X$ ,  $\psi(x)$  witnesses the fact that the countable family  $\phi([x]_E)$  does not have property  $P$ . However,

if we require  $\psi$  to be a Borel function which does not depend essentially on the representative in  $[x]_E$ , then such a function may or may not exist, depending on  $E$ . Thus for each cardinal invariant whose definition fits our framework, we obtain a new “Borel cardinal invariant property” that corresponds to the given cardinal in the same way that Borel boundedness corresponds to  $\mathfrak{b}$ . Our goal in this paper is to introduce these new Borel cardinal invariant notions and investigate their basic properties.

This paper will be organized as follows. We first recall some basic facts about countable Borel equivalence relations in Section 2, and then in Section 3 we consider the unions problem and its connection to Borel boundedness in greater detail. In Section 4, we introduce a slew of Borel cardinal invariant properties derived from familiar cardinal invariants of the continuum. For the sake of the general theory, we propose a slight alteration in the definition of Borel boundedness that we show to be equivalent to the one introduced in [BJ07]. In Section 5, we situate the preceding discussion in the abstract setting of relations and morphisms as developed by Vojtáš and Blass, and prove some general results concerning the Borel cardinal invariant properties that correspond to the so-called “tame” cardinal invariants. We also show that Thomas’s argument that Martin’s Conjecture implies there exist non-hyperfinite, non-Borel bounded relations can be generalized to a large class of Borel cardinal invariant properties. In Section 6, we discuss the special case of the splitting number  $\mathfrak{s}$ , and show that its corresponding Borel cardinal invariant property coincides with smoothness. Finally, in Section 7, we establish a Hasse diagram of inequalities between the Borel cardinal invariant properties, and make some conjectures about additional implications that we are currently unable to establish.

We wish to thank Simon Thomas for originally suggesting this project.

## 2. PRELIMINARIES

In this section, we recall some basic facts and definitions from the theory of Borel equivalence relations.

A *standard Borel space* is a measurable space  $(X, \mathcal{B})$  such that  $\mathcal{B}$  arises as the Borel  $\sigma$ -algebra of some Polish topology on  $X$ . Here a topological space is *Polish* if it admits a complete, separable metric. If  $A$  is any countable set, then the product of discrete topologies on  $A^\omega$  is Polish. In particular, Cantor space  $2^\omega$  and Baire space  $\omega^\omega$  are Polish, as is  $\mathbb{R}$  in the usual topology. The set  $[\omega]^\omega$  of infinite subsets of  $\omega$  can be viewed as a Borel subset of  $2^\omega$ , and hence as a standard Borel space in its own right. The appropriate notion of isomorphism in the context of standard Borel spaces is bimeasurable bijection, which we call *Borel isomorphism*. By a classical result, any two uncountable standard Borel spaces

are Borel isomorphic. This will allow us to view the standard Borel spaces  $\mathbb{R}$ ,  $2^\omega$ ,  $\omega^\omega$ , and  $[\omega]^\omega$  as equivalent, so that we may work on whichever is most convenient.

If  $X = (X, \mathcal{B})$  is a standard Borel space, then an equivalence relation  $E$  on  $X$  is called *Borel* if  $E$  is Borel as a subset of  $X \times X$ . The Borel equivalence relation  $E$  is called *countable* if each of its equivalence classes is countable, *finite* if each  $E$ -class is finite, and *hyperfinite* if  $E$  can be expressed as a countable increasing union  $E = \bigcup_n F_n$  of finite Borel equivalence relations  $F_n$ .

Let  $E$  and  $F$  be Borel equivalence relations on the standard Borel spaces  $X$  and  $Y$ , respectively. A function  $f: X \rightarrow Y$  is *Borel* if the graph of  $f$  is Borel as a subset of  $X \times Y$ , or equivalently if  $f^{-1}(B)$  is Borel in  $X$  whenever  $B$  is Borel in  $Y$ . A *Borel homomorphism* from  $E$  to  $F$  is a Borel function  $f: X \rightarrow Y$  such that  $x E y$  implies  $f(x) F f(y)$  for all  $x, y \in X$ . If the Borel function  $f: X \rightarrow Y$  has the stronger property that

$$x E y \iff f(x) F f(y)$$

for every  $x, y \in X$ , then  $f$  is called a *Borel reduction* from  $E$  to  $F$ . If there exists a Borel reduction from  $E$  to  $F$ , then we say that  $E$  is *Borel reducible* to  $F$ , and write  $E \leq_B F$ . We say that  $E$  and  $F$  are *Borel bireducible*, and write  $E \sim_B F$ , if both  $E \leq_B F$  and  $F \leq_B E$ . We call an injective Borel reduction an *embedding*, and write  $E \sqsubseteq_B F$  if  $E$  embeds into  $F$ . Finally, we say that  $E$  and  $F$  are *Borel isomorphic* and write  $E \cong_B F$  if there exists a bijective Borel reduction from  $E$  to  $F$ .

Let  $E$  be a Borel equivalence relation on the standard Borel space  $X$ . Then  $E$  is said to be *smooth* if there is a standard Borel space  $Y$  such that  $E \leq_B \Delta(Y)$ , where  $\Delta(Y)$  denotes the identity relation on  $Y$ . If  $E$  has uncountably many classes, then  $\Delta(\mathbb{R}) \leq_B E$  by the Silver dichotomy. A countable Borel equivalence relation  $E$  is smooth if and only if it admits a Borel transversal, *i.e.*, a Borel set  $B \subset X$  such that  $|B \cap [x]_E| = 1$  for every  $x \in X$ . Every finite Borel equivalence relation is smooth.

Many of the equivalence relations that we consider arise from Borel actions of countable groups. If  $\Gamma$  is a countable group and  $X$  is a standard Borel space, then we endow  $\Gamma$  with the discrete topology and say that the action of  $\Gamma$  on  $X$  is *Borel* if it is Borel as a function  $\Gamma \times X \rightarrow X$ , or equivalently if for each  $\gamma \in \Gamma$ , the map  $x \mapsto \gamma x$  is Borel. In this case we write  $E_\Gamma^X$  for the induced orbit equivalence relation defined by

$$x E_\Gamma^X y \iff (\exists \gamma \in \Gamma) y = \gamma x.$$

Clearly  $E_\Gamma^X$  is countable Borel. Conversely, by a remarkable result of Feldman and Moore, if  $E$  is an arbitrary countable Borel equivalence relation on the standard Borel space  $X$ ,

then there exists a countable group  $\Gamma$  and a Borel action of  $\Gamma$  on  $X$  such that  $E = E_\Gamma^X$ . We shall make use of this representation theorem for countable Borel equivalence relations quite frequently.

For elements  $\alpha, \beta$  of either Cantor space  $2^\omega$  or Baire space  $\omega^\omega$ , let  $\alpha =^* \beta$  if  $\alpha(n) = \beta(n)$  for all but finitely many  $n$ , and let  $\alpha \leq^* \beta$  if  $\alpha(n) \leq \beta(n)$  for all but finitely many  $n$ . Then  $=^*$  is a Borel equivalence relation on  $2^\omega$ , which we call *eventual equality* and denote  $E_0$ . The eventual equality relations on  $\omega^\omega$  and on  $[\omega]^\omega$  are Borel isomorphic to  $E_0$ , so we shall not hesitate to refer to them also as  $E_0$ , where no confusion is possible.  $E_0$  is the unique non-smooth hyperfinite Borel equivalence relation up to Borel bireducibility, and by the general Glimm-Effros dichotomy, every nonsmooth Borel equivalence relation on a standard Borel space embeds  $E_0$ . For a survey of the general theory of the hyperfinite Borel equivalence relations, see [DJK94] or [JKL02].

Finally, we include a consequence of the Lusin-Novikov small sections uniformization theorem [Kec95, 18.10] that we shall require.

**2.1. Proposition.** *Suppose  $X$  and  $Y$  are standard Borel spaces, and suppose that  $f: X \rightarrow Y$  is a countable-to-one Borel function. Then  $\text{im}(f)$  is a Borel subset of  $Y$ , and there exists a Borel function  $\sigma: \text{im}(f) \rightarrow X$  such that  $f \circ \sigma = \text{id}_{\text{im}(f)}$ .*

**2.2. Remark.** The Borel function  $\sigma$  given in Proposition 2.1 is called a *Borel section* for  $f$ . If  $E$  and  $F$  are countable Borel equivalence relations with  $E \leq_B F$ , then any Borel reduction  $f$  from  $E$  to  $F$  is countable-to-one, and hence admits a Borel section  $\sigma: \text{im}(f) \rightarrow X$ .

### 3. BOREL BOUNDEDNESS AND THE UNIONS PROBLEM

Question 1.1 is perhaps the most basic open problem in the study of hyperfinite Borel equivalence relations. In this section we give an “honest attempt” to answer it, and observe how such an attempt leads naturally to the notion of Borel boundedness.

Thus for each  $n, m \in \mathbb{N}$ , let  $E_n^m$  be a finite Borel equivalence relation on the standard Borel space  $X$ , and let  $E_n = \bigcup_m E_n^m$  and  $E = \bigcup_n E_n$  be increasing unions, so that each  $E_n$  is hyperfinite and  $E$  is hyper-hyperfinite. This can be pictured as the infinite grid in Figure 1. Further assume that  $E$  is the orbit equivalence relation arising from the Borel action of the countable group  $\Gamma = \{\gamma_i : i \in \omega\}$ , with  $\gamma_0 = \text{id}$ .

As a Borel subset of  $X \times X$ ,  $E$  is itself a standard Borel space. Define the Borel function  $\chi_E : E \rightarrow \omega^\omega$  by

$$\chi_E(x, y)(n) := \begin{cases} \text{the least } m \text{ such that } x E_n^m y & \text{if } x E_n y \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{array}{rcl}
E_0 & = & E_0^0 \cup E_0^1 \cup E_0^2 \cup \\
& \cap & \\
E_1 & = & E_1^0 \cup E_1^1 \cup E_1^2 \cup \dots \\
& \cap & \\
E_2 & = & E_2^0 \cup E_2^1 \cup E_2^2 \cup \\
& \vdots &
\end{array}$$

FIGURE 1. A hyper-hyperfinite relation. Here, we mean for the rows to be increasing unions:  $E_n^m \subset E_n^{m+1}$ .

Then  $\chi_E$  catalogues exactly when a pair of elements  $(x, y) \in E$  becomes equivalent in each union  $E_n = \bigcup E_n^m$  for which  $x E_n y$ .

Now, let us attempt to solve the unions problem by expressing  $E$  as a countable increasing union  $E = \bigcup_k F_k$  of finite Borel equivalence relations  $F_k$ . Naively, one might simply try to “choose the right sequence” through Figure 1, and set each  $F_k$  equal to an appropriate  $E_n^m$ , moving down and to the right as  $k$  increases. Of course, this is bound to fail, since there is no reason a relation  $E_n^m$  in row  $n$  should contain some  $E_{n'}^{m'}$  in row  $n'$  just because  $n > n'$ . In fact, it is not too difficult to construct a hyper-hyperfinite Borel equivalence relation  $E = \bigcup_n \bigcup_m E_n^m$  such that for all  $n$  and  $m$  and for all  $k \neq n$ ,  $E_n^0 \not\subset E_k^m$ .<sup>1</sup>

To ensure that the  $F_k$ 's are increasing, as a next step one might try taking them to be unions or intersections of the  $E_n^m$ . Of course, the union of two equivalence relations need not be transitive, and there is no reason for the transitive closure of the union of two finite equivalence relations to be finite; so we are led to take intersections of the  $E_n^m$ . As each row is increasing, we need only take one from each row. Since the union of the  $F_k$ 's must exhaust  $E$ , we should start deleting rows from the intersection as  $k$  increases. This suggests that we let

$$F_k := \bigcap_{n \geq k} E_n^{\psi(n)}$$

for some sequence of choices  $\psi \in \omega^\omega$ .

<sup>1</sup>For instance, let  $E$  be the trivial relation on  $\mathbb{N} \times \mathbb{N}$  and let:

$$\begin{aligned}
(i, j) E_n (i', j') &\iff (i, j) = (i', j') \text{ or } i, i' \leq n \\
(i, j) E_n^m (i', j') &\iff (i, j) = (i', j') \text{ or } i = i' = n \text{ or } i, i' < n \wedge j, j' < m.
\end{aligned}$$

All that is left now is to make sure that the union exhausts  $E$ . For this we need precisely the following condition on  $\psi$ : for all  $(x, y) \in E$ , there exists  $n \in \omega$  such that for every  $k \geq n$ ,  $\psi(k) \geq \chi_E(x, y)(k)$ . In other words, we simply need

$$(\forall z \in E) \chi_E(z) \leq^* \psi.$$

Naturally we cannot ask  $\psi$  to dominate an entire family of functions of size continuum; therefore, we should allow  $\psi$  to depend on the equivalence class, so that each  $\psi([x]_E)$  only has to dominate countably many functions at a time. Since we want the  $F_k$  to be Borel, this dependence will have to be Borel as well. Thus we require a Borel,  $E$ -invariant function  $\psi: X \rightarrow \omega^\omega$  such that for each  $(x, y) \in E$ ,

$$\chi_E(x, y) \leq^* \psi(x) = \psi(y).$$

If we define  $\phi_0: X \rightarrow \omega^\omega$  by

$$\phi_0(x)(n) := \max_{i \leq n} \chi_E(x, \gamma_i x),$$

then clearly  $\chi_E(x, y) \leq^* \phi_0(x)$  for all  $x \in X$ , and so it will suffice to ask that for each  $x \in X$ ,

$$(3.1) \quad \phi_0(x) \leq^* \psi(x).$$

Should we simply ask for  $\psi$  to dominate *every* Borel function  $\phi: X \rightarrow \omega^\omega$ , we arrive at the definition of *invariant Borel boundedness*.

**3.2. Definition.** Let  $E$  be a countable Borel equivalence relation on the standard Borel space  $X$ . Then  $E$  is *invariantly Borel bounded* if for every Borel function  $\phi: X \rightarrow \omega^\omega$ , there exists an  $E$ -invariant Borel function  $\psi: X \rightarrow \omega^\omega$  such that  $\phi(x) \leq^* \psi(x)$  for all  $x \in X$ .

This property is too strong, however, as it is easily seen to be equivalent to smoothness.

**3.3. Proposition.** Let  $E$  be a countable Borel equivalence relation on the standard Borel space  $X$ . Then  $E$  is smooth if and only if  $E$  is invariantly Borel bounded.

*Proof.* Let  $E = E_X^\Gamma$  be the orbit equivalence relation arising from the Borel action of the countable group  $\Gamma = \{\gamma_i : i \in \omega\}$  on  $X$ , where  $\gamma_0 = \text{id}$ . Suppose that  $E$  is smooth, and let  $B \subset X$  be a Borel transversal for  $E$ . Define the Borel function  $\sigma: X \rightarrow X$  so that for all  $x \in X$ ,  $\sigma(x)$  is the unique element  $y \in B$  such that  $x E y$ . Let  $\phi: X \rightarrow \omega^\omega$  be an arbitrary Borel function. Then we may define the function  $\psi: X \rightarrow \omega^\omega$  by

$$\psi(x)(n) := \max_{i \leq n} \{\phi(\gamma_i \sigma(x))(n)\}.$$

Clearly  $\psi$  is Borel and  $E$ -invariant, and  $\phi(x) \leq^* \psi(x)$  for all  $x \in X$ .

For the converse, we show that  $E_0$  is *not* invariantly Borel bounded; the result will then follow from the fact, proved below in the remark immediately following Proposition 3.5, that invariant Borel boundedness is closed under Borel reducibility. Thus suppose for contradiction that  $E_0$  is invariantly Borel bounded. Identify each  $x \in 2^\omega$  with the corresponding subset of  $\omega$ , and define  $\tau(x)$  to be the increasing enumeration of  $x$  if  $x$  is infinite, and constantly zero otherwise, so that  $\tau(x) \in \omega^\omega$ . Let  $\psi: 2^\omega \rightarrow \omega^\omega$  be an  $E_0$ -invariant Borel function such that  $\tau(x) \leq^* \psi(x)$  for all  $x \in 2^\omega$ . Let  $D \subset 2^\omega$  be a comeager subset on which  $\psi$  is continuous, so that

$$\hat{D} := \bigcap_{\gamma \in \Gamma} \gamma D$$

is comeager and  $E_0$ -invariant. Fix  $x_0 \in \hat{D}$ . Since  $[x_0]_{E_0}$  is dense and  $\psi$  is constant on  $[x_0]_{E_0}$ , by continuity of  $\psi$  we have  $\psi(x) = \psi(x_0)$  for all  $x \in \hat{D}$ . However, the set

$$\{x \in 2^\omega : \tau(x) \not\leq^* \psi(x_0)\}$$

is clearly comeager, a contradiction.  $\square$

In Equation 3.1, we required only that  $\psi(x)$  *eventually* dominate  $\phi_0(x)$  for each  $x \in X$ , so there is no reason to insist on  $\psi$  being  $E$ -invariant; rather, it will suffice to have  $\psi(x) =^* \psi(y)$  whenever  $x E y$ , and hence it is more natural to ask for  $\psi$  to be quasi-invariant. This leads to the definition introduced by Boykin and Jackson in [BJ07].

**3.4. Definition.** Let  $E$  be a countable Borel equivalence relation on the standard Borel space  $X$ . Then  $E$  is *Borel bounded* if for every Borel function  $\phi: X \rightarrow \omega^\omega$ , there exists a Borel homomorphism  $\psi: X \rightarrow \omega^\omega$  from  $E$  to  $=^*$  such that for all  $x \in X$ ,  $\phi(x) \leq^* \psi(x)$ .

This definition is nontrivial, since for instance any hyperfinite relation is Borel bounded. Indeed, suppose that  $E = \bigcup_n F_n$  is hyperfinite, with each  $F_n$  finite. Then given any Borel function  $\phi: X \rightarrow \omega^\omega$ , we can define

$$\psi(x)(n) := \max \{ \phi(y)(n) : y F_n x \} ,$$

so that  $\psi$  is a Borel homomorphism from  $E$  to  $=^*$  such that  $\phi(x) \leq^* \psi(x)$  for all  $x \in X$ .

As we saw above, Borel boundedness is tailor-made for obtaining Theorem 1.4, and we now present the proof.



*Proof of Theorem 1.4.* Let  $E = \bigcup_n E_n$  be the increasing union of hyperfinite equivalence relations  $E_n$ , where for each  $n$ ,  $E_n = \bigcup_m E_n^m$  is the increasing union of finite Borel equivalence relations  $E_n^m$ . Also, let  $E$  be the orbit equivalence relation induced by the action of the countable group  $\Gamma = \{\gamma_i : i \in \omega\}$ , where  $\gamma_0 = \text{id}$ .

Now define  $\phi_0$  as in Equation 3.1, so that for each  $x, n$ ,  $\phi_0(x)(n)$  is the least  $m$  such that:

- whenever  $y \in \{\gamma_0 x, \dots, \gamma_n x\}$  and  $y E_n x$ , then in fact  $y E_n^m x$ .

Since  $E$  is Borel bounded, there exists a Borel homomorphism  $\psi: E \rightarrow =^*$  such that for all  $x \in X$ ,  $\phi_0(x) \leq^* \psi(x)$ . We may therefore define  $F_n$  by:

- $x F_n y$  iff for all  $k \geq n$ , we have  $\psi(x)(k) = \psi(y)(k)$  and  $x E_k^{\psi(x)(k)} y$ .

(The “for all  $k \geq n$ ” is needed to make the  $F_n$  increasing, the “ $\psi(x)(k) = \psi(y)(k)$ ” is needed to make  $F_n$  symmetric, the “ $E_k$ ” is needed to make  $F_n$  finite, and the “ $\psi(x)(k)$ ” is needed to ensure the  $F_n$  will exhaust  $E$ .)

It is clear that  $\langle F_n \rangle$  is an increasing sequence of finite equivalence relations contained in  $E$ . The last thing to check is that  $E = \bigcup_n F_n$ . Indeed, if  $x E y$ , then write  $y = \gamma_i x$  and choose some  $n > i$  such that for all  $k \geq n$ , we have  $x E_k y$  and

$$\max \{ \phi_0(x)(k), \phi_0(y)(k) \} \leq \psi(x)(k) = \psi(y)(k) .$$

Then by definition of  $\phi_0$ , for each  $k \geq n$  we have  $x E_k^{\phi_0(x)(k)} y$  and hence  $x E_k^{\psi(x)(k)} y$ .  $\square$

As mentioned in the introduction, it is not known whether there exist Borel bounded countable Borel equivalence relations that are not hyperfinite.

We conclude this section with a proof of the fact that Borel boundedness is closed under Borel reducibility. This result is Lemma 10 of [BJ07], but we present a proof that is designed to motivate our subsequent discussion about the definitions of other Borel invariant properties.

**3.5. Proposition.** *Let  $E$  and  $F$  be countable Borel equivalence relations on the standard Borel spaces  $X$  and  $Y$ , respectively. If  $E \leq_B F$  and  $F$  is Borel bounded, then  $E$  is Borel bounded.*

*Proof.* Let  $\phi: X \rightarrow \omega^\omega$  be any Borel function. Suppose that  $f: X \rightarrow Y$  is a Borel reduction from  $E$  to  $F$ , and define the equivalence relation  $E' \subset E$  on  $X$  by  $x E' y$  iff  $f(x) = f(y)$ . Then  $E'$  is smooth, and therefore Borel bounded. Let  $\phi': X \rightarrow \omega^\omega$  be a Borel homomorphism from  $E'$  to  $=^*$  such that for all  $x \in X$ ,  $\phi(x) \leq^* \phi'(x)$ . Also let  $B \subset X$  be a Borel transversal for  $E'$ , with  $\sigma: \text{im}(f) \rightarrow X$  a Borel function such that  $f \circ \sigma = \text{id}_{\text{im}(f)}$  (see figure 2). Now define the Borel function  $\tilde{\phi}: Y \rightarrow \omega^\omega$  by

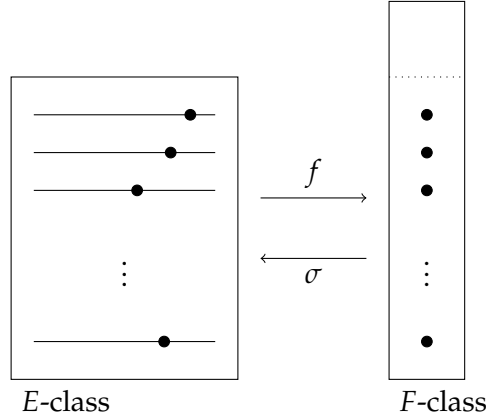


FIGURE 2.  $\sigma$  gives us a base point with which to order each fiber of  $f$ .

$$\tilde{\phi}(y)(n) := \begin{cases} \phi'(\sigma(y))(n) & \text{if } y \in \text{im}(f) \\ 0 & \text{otherwise .} \end{cases}$$

Using the fact that  $F$  is Borel bounded, let  $\tilde{\psi}: Y \rightarrow \omega^\omega$  be a Borel homomorphism from  $F$  to  $=^*$  such that for all  $y \in Y$ ,  $\tilde{\phi}(y) \leq^* \tilde{\psi}(y)$ . Finally, let  $\psi = \tilde{\psi} \circ f$ . It is easily checked that  $\psi: X \rightarrow \omega^\omega$  is a Borel homomorphism from  $E$  to  $=^*$  such that for all  $x \in X$ ,  $\phi(x) \leq^* \psi(x)$ .  $\square$

In the proof of Proposition 3.3, we required that *invariant* Borel boundedness is also closed downward under Borel reducibility. Indeed, this follows using the same argument, since if  $F$  is invariantly Borel bounded, then the Borel function  $\tilde{\psi}$  in the proof of Proposition 3.5 can be chosen to be  $F$ -invariant, in which case  $\psi$  will be  $E$ -invariant.

#### 4. BOREL CARDINAL INVARIANT PROPERTIES

In the definition of Borel boundedness, the Borel function  $\phi$  assigns a countable family of elements of  $\omega^\omega$  to each  $E$ -class  $[x]_E$ . Since no countable family is unbounded, there is always a witness  $\psi(x)$  which bounds  $\phi([x]_E)$ . Borel boundedness means that this witness can be chosen in an explicit and quasi-invariant fashion. An analogous definition can be made in which unbounded families are replaced by other types of families which appear in the study of cardinal invariants of the continuum: splitting families, maximal almost disjoint families, ultrafilter bases, and so on. We will now introduce Borel cardinal invariant properties that correspond to these other cardinal invariants of the continuum in the same way that Borel boundedness corresponds to the bounding number,  $\mathfrak{b}$ .

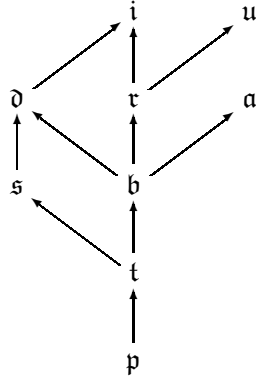


FIGURE 3. Size relationships among several cardinal invariants

We focus in this section on a few important cardinal invariants which are combinatorial in nature. The relationship between the sizes of these cardinals is described visually by (a subset of) the so-called van Douwen diagram, which appears in Figure 3. The article [Bla03] gives a full discussion of these cardinals and their relationships. As motivating examples we consider the splitting number  $\mathfrak{s}$  and the pseudo-intersection number  $\mathfrak{p}$ .

Given a subset  $A \subset \omega$ , write  $A^c = \omega \setminus A$ . Given sets  $A, B \subset \omega$ , we say that  $A$  *splits*  $B$  if  $|A \cap B| = |A^c \cap B| = \aleph_0$ . A family  $\mathcal{S} \subset [\omega]^\omega$  of infinite subsets of  $\omega$  is a *splitting family* if for every infinite set  $B \subset \omega$  there exists  $A \in \mathcal{S}$  such that  $A$  splits  $B$ . The *splitting number*  $\mathfrak{s}$  is defined to be the minimum cardinality of a splitting family.

A family  $\mathcal{F} \subset [\omega]^\omega$  of infinite subsets of  $\omega$  is *centered* if every finite subfamily of  $\mathcal{F}$  has infinite intersection. The infinite set  $A \subset \omega$  is said to be a *pseudo-intersection* of the centered family  $\mathcal{F}$  if  $A \subset^* B$  for every  $B \in \mathcal{F}$ . The pseudo-intersection number  $\mathfrak{p}$  is defined to be the minimum cardinality of a centered family with no pseudo-intersection.

As it will be relevant later, we briefly sketch a proof of the fact that no countable family of subsets of  $\omega$  can be a splitting family.

#### 4.1. Proposition. $\aleph_0 < \mathfrak{s}$ .

*Proof.* Let  $\{A_n : n \in \omega\}$  be a countable family of subsets of  $\omega$ . Given some nonprincipal ultrafilter  $\mathcal{U}$  on  $\omega$ , we can set  $B_{-1} = \omega$  and then inductively define  $B_{n+1}$  to be whichever of the sets  $B_n \cap A_n$ ,  $B_n \cap A_n^c$  is in  $\mathcal{U}$ . Then each  $B_n$  is infinite, so if we inductively choose  $b_{n+1} \in B_{n+1}$  distinct from  $b_0, \dots, b_n$ , then  $\{b_n : n \in \omega\}$  is not split by any  $A_n$ .  $\square$

It follows that if  $E$  is a Borel equivalence relation on the standard Borel space  $X$ , then for each Borel function  $\phi: X \rightarrow [\omega]^\omega$ , there trivially exists an  $E$ -invariant function  $\psi: X \rightarrow$

$[\omega]^\omega$  such that for each  $x \in X$ ,  $\psi(x)$  witnesses the fact that  $\phi([x]_E)$  is not a splitting family; i.e., such that for each  $x \in X$ ,  $\psi(x) \subset^* \phi(x)$  or  $\psi(x) \subset^* \phi(x)^c$ . In analogy with Borel boundedness, we might therefore call  $E$  *Borel non-splitting* if such a function  $\psi$  can be chosen in an explicit and quasi-invariant fashion.

**4.2. Definition** (temporary). Let  $E$  be a countable Borel equivalence relation on the standard Borel space  $X$ . Then  $E$  is *Borel non-splitting* if for every Borel function  $\phi: X \rightarrow [\omega]^\omega$ , there exists a Borel homomorphism  $\psi: X \rightarrow [\omega]^\omega$  from  $E$  to  $=^*$  such that for all  $x \in X$ ,  $\psi(x) \subset^* \phi(x)$  or  $\psi(x) \subset^* \phi(x)^c$ .

Similarly, every countable centered family  $\mathcal{F} \subset [\omega]^\omega$  has a pseudo-intersection; hence  $\aleph_0 < \mathfrak{p}$ , which suggests the following definition.

**4.3. Definition** (temporary). Let  $E$  be a countable Borel equivalence relation on the standard Borel space  $X$ . Then  $E$  is *Borel pseudo-intersecting* if for every Borel function  $\phi: X \rightarrow [\omega]^\omega$  such the family  $\{\phi(y) : y E x\}$  is centered for all  $x \in X$ , there exists a Borel homomorphism  $\psi: X \rightarrow [\omega]^\omega$  from  $E$  to  $=^*$  such that for all  $x \in X$ ,  $\psi(x) \subset^* \phi(x)$ .

If these are to be reasonable properties of countable Borel equivalence relations, then it is highly desirable for them to be closed under Borel reducibility, or at least to be  $\sim_B$ -invariant. However, two different problems arise if one attempts to prove the analogue of Proposition 3.5 for these notions as currently defined.

Notice that the proof of Proposition 3.5 involved two separate diagonalizations. First, each fiber  $f^{-1}(\{y\})$  yielded a countable family  $\{\phi(x) : f(x) = y\} \subset \omega^\omega$  that was dominated by  $\phi'(\sigma(y))$ ; then the countable family  $\{\phi'(\sigma(z)) : z \in [y]_F\}$  was dominated by  $\tilde{\psi}(y)$ , which served to dominate the entire original family  $\{\phi(x) : x E \sigma(y)\}$ . Letting  $\mathfrak{x}$  (that's an 'x') be the cardinal under consideration, it is apparent that this argument only works if the property of "witnessing that  $\aleph_0 < \mathfrak{x}$ " is transitive. It clearly is in the case of  $\mathfrak{b}$ , but if each  $B_n \subset \omega$  witnesses that  $\{A_n^m : m \in \omega\}$  is not splitting, and if  $C \subset \omega$  witnesses that  $\{B_n : n \in \omega\}$  is not splitting, then nevertheless  $C$  may very well be split by some set  $A_n^m$ .

The pseudo-intersecting property yields an even more fundamental problem. If

$$\{A_n^m : m, n \in \omega\}$$

is a centered family of subsets of  $\omega$ , and if each  $B_n$  is a pseudo-intersection of  $\{A_n^m : m \in \omega\}$ , then there is no reason for  $\{B_n : n \in \omega\}$  even to be centered, so there is no way to carry out a second diagonalization.

Consequently we propose the following slight adjustment in the definition of Borel boundedness, justified below by Proposition 4.5, so that it can properly serve as a model

for the general theory. Recall that if  $X$  is a standard Borel space, then we define the equivalence relation  $E_{\text{set}}(X)$  on  $X^\omega$  by

$$\langle x_n \rangle E_{\text{set}}(X) \langle x'_n \rangle \iff \{x_n : n \in \omega\} = \{x'_n : n \in \omega\} .$$

If  $X$  is clear from context, we shall often write  $E_{\text{set}}$  instead of  $E_{\text{set}}(X)$ .

**4.4. Definition.** Let  $E$  be a countable Borel equivalence relation on the standard Borel space  $X$ . Then  $E$  has property **b** ( $E$  is *Borel bounded*) if for every Borel homomorphism  $\phi: X \rightarrow (\omega^\omega)^\omega$  from  $E$  to  $E_{\text{set}}(\omega^\omega)$ , there exists a Borel homomorphism  $\psi: X \rightarrow \omega^\omega$  from  $E$  to  $=^*$  such that for all  $x \in X$  and for all  $n \in \omega$ ,  $\phi(x)(n) \leq^* \psi(x)$ .

In the original Definition 3.4, the function  $\phi$  assigns a countable family of functions  $\phi([x]_E) \subset \omega^\omega$  to each  $E$ -class by associating a single one to each element in the class. In Definition 4.4, we give *each* element  $x \in X$  knowledge of the entire family of functions that  $\phi$  associates to  $[x]_E$ . As we will see below, this slight tweak will enable us to prove closure under  $\leq_B$  for whatever Borel invariant property we choose to consider.

**4.5. Proposition.** Let  $E$  be a countable Borel equivalence relation on the standard Borel space  $X$ . Then  $E$  is Borel bounded in the sense of Definition 3.4 if and only if  $E$  has property **b**, i.e., is Borel bounded in the sense of Definition 4.4.

*Proof.* Let  $E$  be the orbit equivalence relation arising from the Borel action of the countable group  $\Gamma = \langle \gamma_n : n \in \omega \rangle$ . Suppose  $E$  has property **b**, and let  $\phi: X \rightarrow \omega^\omega$  be a Borel function. Then for each  $x \in X$  and  $n \in \omega$ , define  $\phi'(x)(n) = \phi(\gamma_n x)$ , so that  $\phi'$  is a Borel homomorphism from  $E$  to  $E_{\text{set}}(\omega^\omega)$ . If the function  $\psi: X \rightarrow \omega^\omega$  is a Borel homomorphism from  $E$  to  $=^*$  such that  $\phi'(x)(n) \leq^* \psi(x)$  for all  $x \in X$  and  $n \in \omega$ , then clearly  $\phi(x) \leq^* \psi(x)$  for all  $x \in X$ .

Conversely, suppose  $E$  is Borel bounded and let  $\phi': X \rightarrow (\omega^\omega)^\omega$  be a Borel homomorphism from  $E$  to  $E_{\text{set}}(\omega^\omega)$ . Then for each  $x \in X$  and  $n \in \omega$ , define

$$\phi(x)(n) := \max_{k \leq n} \{ \phi'(x)(k)(n) \} ,$$

so that  $\phi: X \rightarrow \omega^\omega$  is a Borel function such that for each  $x \in X$  and  $n \in \omega$ ,  $\phi'(x)(n) \leq^* \phi(x)$ . Obtain a Borel homomorphism  $\psi: X \rightarrow \omega^\omega$  from  $E$  to  $=^*$  such that  $\phi(x) \leq^* \psi(x)$  for all  $x \in X$ . Clearly  $\phi'(x)(n) \leq^* \psi(x)$  for each  $x \in X$  and  $n \in \omega$ .  $\square$

We are now ready to introduce a zoo of “Borel cardinal invariant properties” of countable Borel equivalence relations, each of which corresponds to a cardinal invariant of the continuum in the same way that Borel boundedness corresponds to **b**. These definitions

will be modeled on Definition 4.4, and in each case depend on the fact that the relevant cardinal invariant is uncountable. We include property b in the list for completeness.

**4.6. Definition.** Let  $E$  be a countable Borel equivalence relation on the standard Borel space  $X$ .

- $E$  has property b ( $E$  is *Borel bounded*) if for every Borel homomorphism  $\phi: X \rightarrow (\omega^\omega)^\omega$  from  $E$  to  $E_{\text{set}}$ , there exists a Borel homomorphism  $\psi: X \rightarrow \omega^\omega$  from  $E$  to  $=^*$  such that for all  $x \in X$  and for all  $n \in \omega$ ,  $\phi(x)(n) \leq^* \psi(x)$ .
- $E$  has property d ( $E$  is *Borel non-dominating*) if for every Borel homomorphism  $\phi: X \rightarrow (\omega^\omega)^\omega$  from  $E$  to  $E_{\text{set}}$ , there exists a Borel homomorphism  $\psi: X \rightarrow \omega^\omega$  from  $E$  to  $=^*$  such that for all  $x \in X$  and for all  $n \in \omega$ ,  $\psi(x) \not\leq^* \phi(x)(n)$ .
- $E$  has property s ( $E$  is *Borel non-splitting*) if for every Borel homomorphism  $\phi: X \rightarrow ([\omega]^\omega)^\omega$  from  $E$  to  $E_{\text{set}}$ , there exists a Borel homomorphism  $\psi: X \rightarrow [\omega]^\omega$  from  $E$  to  $=^*$  such that for all  $x \in X$  and for all  $n \in \omega$ ,  $\psi(x) \subset^* \phi(x)(n)$  or  $\psi(x) \subset^* \phi(x)(n)^c$ .
- $E$  has property r ( $E$  is *Borel reaping*) if for every Borel homomorphism  $\phi: X \rightarrow ([\omega]^\omega)^\omega$  from  $E$  to  $E_{\text{set}}$ , there exists a Borel homomorphism  $\psi: X \rightarrow [\omega]^\omega$  from  $E$  to  $=^*$  such that for all  $x \in X$  and for all  $n \in \omega$ ,  $|\psi(x) \cap \phi(x)(n)| = |\psi(x)^c \cap \phi(x)(n)| = \aleph_0$ .
- $E$  has property p ( $E$  is *Borel pseudo-intersecting*) if for every Borel homomorphism  $\phi: X \rightarrow ([\omega]^\omega)^\omega$  from  $E$  to  $E_{\text{set}}$  such that  $\{\phi(x)(n) : n \in \omega\}$  is centered for each  $x \in X$ , there exists a Borel homomorphism  $\psi: X \rightarrow [\omega]^\omega$  from  $E$  to  $=^*$  such that for every  $x \in X$  and  $n \in \omega$ ,  $\psi(x) \subset^* \phi(x)(n)$ .
- $E$  has property t ( $E$  is *Borel tower-plugging*) if for every Borel homomorphism  $\phi: X \rightarrow ([\omega]^\omega)^\omega$  from  $E$  to  $E_{\text{set}}$  such that  $\{\phi(x)(n) : n \in \omega\}$  admits a well-ordering compatible with  $\subset^*$ , there exists a Borel homomorphism  $\psi: X \rightarrow [\omega]^\omega$  from  $E$  to  $=^*$  such that for every  $x \in X$  and  $n \in \omega$ ,  $\psi(x) \subset^* \phi(x)(n)$ .
- $E$  has property a ( $E$  is *Borel non-mad*) if for every Borel homomorphism  $\phi: X \rightarrow ([\omega]^\omega)^\omega$  from  $E$  to  $E_{\text{set}}$  such that  $\{\phi(x)(n) : n \in \omega\}$  is almost disjoint for each  $x \in X$ , there exists a Borel homomorphism  $\psi: X \rightarrow [\omega]^\omega$  from  $E$  to  $=^*$  such that for every  $x \in X$  and  $n \in \omega$ ,  $|\psi(x) \cap \phi(x)(n)| < \aleph_0$ .
- $E$  has property i ( $E$  is *Borel non-maximal independent*) if for every Borel homomorphism  $\phi: X \rightarrow ([\omega]^\omega)^\omega$  from  $E$  to  $E_{\text{set}}$  such that  $\{\phi(x)(n) : n \in \omega\}$  is independent for each  $x \in$

$X$ , there exists a Borel homomorphism  $\psi: X \rightarrow [\omega]^\omega$  from  $E$  to  $=^*$  such that for every  $x \in X$ ,  $\{\psi(x)\} \cup \{\phi(x)(n) : n \in \omega\}$  is independent and  $\psi(x) \notin \{\phi(x)(n) : n \in \omega\}$ .

- $E$  has property  $u$  if for every Borel homomorphism  $\phi: X \rightarrow ([\omega]^\omega)^\omega$  from  $E$  to  $E_{\text{set}}$  such that  $\{\phi(x)(n) : n \in \omega\}$  is centered for each  $x \in X$ , there exists a Borel homomorphism  $\psi: X \rightarrow [\omega]^\omega$  from  $E$  to  $=^*$  such that for every  $x \in X$  and  $n \in \omega$ ,  $|\psi(x) \cap \phi(x)(n)| = |\psi(x)^c \cap \phi(x)(n)| = \aleph_0$ .

Thus we obtain Borel invariant properties for each of the “quotable” cardinal invariants  $\mathfrak{b}$ ,  $\mathfrak{d}$ ,  $\mathfrak{s}$ ,  $\mathfrak{r}$ ,  $\mathfrak{p}$ ,  $\mathfrak{a}$ ,  $\mathfrak{i}$ ,  $\mathfrak{t}$ , and  $u$ , which we name with the same letters in a sans-serif font for convenience. It is easy to show that each of these properties is closed under containment and Borel reducibility. This is proved generally in the next section, but we sketch here the proof for property  $\mathfrak{p}$  to illustrate the motivation behind the use of  $E_{\text{set}}$  in Definition 4.6.

**4.7. Proposition.** *Let  $E$  and  $F$  be countable Borel equivalence relations on the standard Borel spaces  $X$  and  $Y$ , respectively. If  $F$  has property  $\mathfrak{p}$  and  $E \leq_B F$ , then  $E$  has property  $\mathfrak{p}$ .*

*Proof.* Let  $E$  be the orbit equivalence relation arising from the Borel action of the countable group  $\Gamma = \{\gamma_i : i \in \omega\}$  on  $X$ . Suppose that  $f: X \rightarrow Y$  is a Borel reduction from  $E$  to  $F$ , and let  $\sigma: \text{im}(f) \rightarrow X$  be a Borel function such that  $f \circ \sigma = \text{id}_{\text{im}(f)}$ , as depicted in Figure 2. Suppose  $\phi: X \rightarrow ([\omega]^\omega)^\omega$  is a Borel homomorphism from  $E$  to  $E_{\text{set}}$  such that for every  $x \in X$ , the family  $\{\phi(x)(n) : n \in \omega\}$  is centered. We shall define another homomorphism  $\tilde{\phi}: Y \rightarrow ([\omega]^\omega)^\omega$  from  $F$  to  $E_{\text{set}}$  with the analogous property. First, let  $n \mapsto \langle n_0, n_1 \rangle$  denote a fixed pairing function. Then define

$$\tilde{\phi}(y)(n) := \phi(\gamma_{n_0}\sigma(y))(n_1) .$$

Clearly  $\tilde{\phi}$  is a Borel homomorphism from  $F$  to  $E_{\text{set}}$  such that  $\{\tilde{\phi}(y)(n) : n \in \omega\}$  is centered for each  $y$ , so by hypothesis there exists a Borel homomorphism  $\tilde{\psi}: Y \rightarrow [\omega]^\omega$  from  $F$  to  $=^*$  such that  $\tilde{\psi}(y) \subset^* \tilde{\phi}(y)(n)$  for all  $y$  and  $n$ . Now define  $\psi = \tilde{\psi} \circ f$ . Since  $\psi(x)$  and  $\tilde{\psi}(x)$  enumerate the same families, we clearly have  $\psi(x) \subset^* \phi(x)(n)$  for all  $n$ .  $\square$

What we have, then, is a large new family of combinatorial properties of countable Borel equivalence relations. We should now like to establish the basic relationships between these properties, and attempt to locate them as far as possible within the hierarchy of countable Borel equivalence relations under the partial (pre)-order of Borel reducibility.

As we shall see, there is a rough correspondence between ZFC-provable inequalities among cardinal invariants of the continuum and implications among the Borel invariant properties we have introduced. For example, it is immediate from the definitions that

every Borel bounded relation is also Borel non-dominating, for the same reason that  $\mathfrak{b} \leq \mathfrak{d}$ . We express this succinctly by writing  $\mathfrak{b} \rightarrow \mathfrak{d}$ . Likewise, the fact that every tower is a centered family shows both that  $\mathfrak{p} \leq \mathfrak{t}$  and also that  $\mathfrak{p} \rightarrow \mathfrak{t}$ .

Figure 4 displays the basic implications between Borel cardinal invariant properties that we shall prove in Section 7. A special case appears to be property  $\mathfrak{s}$ , which we show in Section 6 to be equivalent to smoothness. At present, most of the implications in Figure 4 are simply due to “obvious” inequalities between cardinal invariants. We hope that the diagram can be improved and expanded in the future.

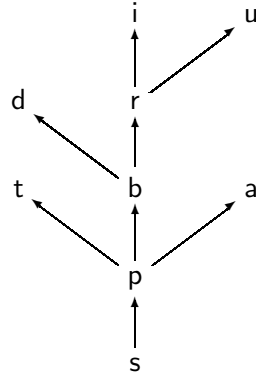


FIGURE 4. Relationships among several Borel invariant properties

It is possible that this forest of definitions will give rise to a varied and interesting family of properties. It is also conceivable that each of these properties implies hyperfiniteness, and hence that the whole diagram collapses. If this is the case, then we will be left with a great number of new characterizations of hyperfiniteness. In light of the challenges in this subject, if the diagram does not collapse then it may be very difficult to prove that this is the case.

We conclude this section by discussing briefly some of the motivations for studying and expanding the diagram in Figure 4, and by considering what meaning the relationships between the Borel invariant properties might have for the cardinals themselves. The diagram may be regarded as being stratified into layers, though we do not know where the boundaries lie at the moment. At the bottom of the diagram we find the characterizations of smoothness, such as property  $\mathfrak{s}$ . Just above that should be the characterizations of (non-smooth) hyperfiniteness. It is not known whether any of the Borel invariant properties lies at this level.



From the perspective of the unions problem, the most interesting layer is the one consisting of those properties  $\times$  that do not imply hyperfiniteness, but additionally have the Boykin-Jackson property that hyper-hyperfiniteness plus  $\times$  imply hyperfiniteness. Once again, we do not know if this layer is nonempty, but it is at this layer that we have a solution for the unions problem.

The top layer consists of those properties which hold of all countable Borel equivalence relations. Once again, we do not know any (nontrivial) Borel invariant property which lies at this layer. But if this layer were to overlap with the previous, then the unions problem would be completely solved.

Finally, we suggest that the Borel invariant properties have another simple meaning. Each one concerns a family of a certain type: a dominating family, a maximal tower, a centered family with no pseudointersection, and so on. None of these families can be countable, as can usually be established by a straightforward diagonalization argument. However, the witness constructed in this diagonalization typically depends on a well-ordering of the family. The corresponding Borel invariant property helps us to gauge the extent of this dependence; the higher the corresponding property lies in the diagram, the easier it is to diagonalize in a way that does not explicitly depend on the well-order.

As an example, consider the diagonalization that is used to prove that no countable family can be unbounded, compared to that used in the proof of Proposition 4.1 to show that no countable family can be a splitting family. The construction given in the latter argument is less explicit and depends to a greater extent on the ordering of the sets  $\langle A_n \rangle$  than that given in Equation 1.3 to show that  $\aleph_0 < \mathfrak{b}$ . In particular, it requires that we make a sequence of dependent and seemingly arbitrary choices, for which we enlisted the aid of a nonprincipal ultrafilter. In Section 6 we demonstrate that this use of an ultrafilter is no accident, and that in fact the diagonalization cannot be carried out in a Borel and (quasi-)invariant fashion when the countable families are given by  $E_0$ -classes.

## 5. BOREL INVARIANT PROPERTIES INSPIRED BY TAME CARDINAL INVARIANTS

Most cardinal invariants of the continuum, including the “quotable” ones considered in Section 4, can be described in a rather dramatically general context as the norm of a suitable relation. See, for instance, Vojtáš [Voj93] and Blass [Bla96] for a development of this abstract approach. We now extend our discussion to this setting, and define the Borel invariant property inspired by nearly any tame cardinal invariant.

Following Vojtáš, let a *relation* be a triple  $A = (A_-, A_+, A)$ , where  $A \subset A_- \times A_+$ . We think of  $A_-$  as a set of “challenges,”  $A_+$  as a set of “responses,” and  $A$  as a “dominating

relation.” We say that a family  $\mathcal{F} \subset A_+$  is *dominating* with respect to the relation  $A$  if for any challenge  $x \in A_-$  there exists a response  $y \in \mathcal{F}$  such that  $x A y$ . Let  $D(A)$  consist of all families  $\mathcal{F} \subset A_+$  that are dominating with respect to  $A$ . Then the *dominating number* or *norm* of the relation  $A$  is the cardinal invariant

$$\|A\| := \min \{|\mathcal{F}| : \mathcal{F} \in D(A)\} .$$

For example, the dominating number  $\mathfrak{d}$  is the norm of the relation  $(\omega^\omega, \omega^\omega, \leq^*)$ , and the splitting number  $\mathfrak{s}$  is the norm of the relation  $([\omega]^\omega, \mathcal{P}(\omega), \text{“is split by”})$ . Indeed, many cardinal invariants can be expressed as the norm of a suitable relation, but a slight generalization is needed to handle certain others. Let  $A$  be a relation, and let  $\Phi$  be a property of families  $\mathcal{F} \subset A_+$ . Then the norm of  $A$  *relative to*  $\Phi$  is the cardinal invariant

$$\|A\|_\Phi := \min \{|\mathcal{F}| : \Phi(\mathcal{F}) \text{ \& } \mathcal{F} \in D(A)\} .$$

For instance, the pseudo-intersection number  $\mathfrak{p}$  is the norm of the relation  $([\omega]^\omega, 2^\omega, \not\leq^*)$  relativized to the property  $\Phi(\mathcal{F}) = \text{“}\mathcal{F} \text{ is centered.”}$

Cardinal invariants that can be expressed as the relativized norm of a relation are said to be *tame*. (This terminology is usually reserved for the case when additional definability constraints are imposed on  $A$  and  $\Phi$ . At present we shall not have need of any further precision, but see Appendix B of [Zap04] for a discussion.) For our purposes, we must consider only those relations  $A$  such that both  $A_-$  and  $A_+$  are subspaces of either  $\omega^\omega$  or  $\mathcal{P}(\omega) = 2^\omega$ . The reason is that on each of these spaces we have an appropriate notion of *almost equality*. More precisely, for  $S \subset 2^\omega$  we let  $E_0(S)$  denote the restriction of  $E_0$  to  $S$ , and for  $S \subset \omega^\omega$  we let  $E_0(S)$  denote the restriction of  $=^*$  to  $S$ . A relation  $A$  is said to be *invariant* if both  $A_-$  and  $A_+$  are Borel subspaces of either  $\omega^\omega$  or  $\mathcal{P}(\omega)$ , and whenever  $x E_0(A_-) x'$  and  $y E_0(A_+) y'$ , we have

$$x A y \iff x' A y' .$$

Now we are ready to define the Borel invariant property *inspired by* the cardinal invariant  $\|A\|_\Phi$  as follows.

**5.1. Definition.** Let  $A$  be an invariant relation, let  $\Phi$  be a property of families  $\mathcal{F} \subset A_+$ , and suppose that  $E$  is a countable Borel equivalence relation on the standard Borel space  $X$ . Then  $E$  is said to be *Borel non-* $(A, \Phi)$  if for every Borel homomorphism  $\phi: X \rightarrow (A_+)^\omega$  from  $E$  to  $E_{\text{set}}(A_+)$  such that the family  $\{\phi(x)(n) : n \in \omega\}$  has property  $\Phi$  for each  $x \in X$ , there exists a Borel homomorphism  $\psi: X \rightarrow A_-$  from  $E$  to  $E_0(A_-)$  such that for all  $x \in X$

and  $n \in \omega$ ,

$$\neg(\psi(x) \mathrel{A} \phi(x)(n)).$$

Borrowing the terminology for cardinal invariants, a Borel invariant property inspired by some  $\|A\|_\Phi$  will be called *tame*, and a Borel invariant property inspired by some  $\|A\|$  will be called *simple*. In the latter case we shall omit  $\Phi$  from the notation and simply say that  $E$  is *Borel non- $A$* .

For example, the Borel property inspired by  $\mathfrak{d} = \|(\omega^\omega, \omega^\omega, \leq^*)\|$  is the Borel non-dominating property,  $\mathfrak{d}$ . Similarly, the Borel pseudo-intersecting property  $\mathfrak{p}$  is the property inspired by the relation  $([\omega]^\omega, \mathcal{P}(\omega), \not\subseteq^*)$  together with  $\Phi(\mathcal{F}) = \text{“}\mathcal{F} \text{ is centered.”}$  In fact, one easily checks that each of the nine Borel invariant properties introduced in Definition 4.6 is tame. Of these, only the four properties  $\mathfrak{b}$ ,  $\mathfrak{d}$ ,  $\mathfrak{r}$ , and  $\mathfrak{s}$  are simple.

We now establish some basic closure results. Recall that if  $E \subset F$  are countable Borel equivalence relations on the standard Borel space  $X$ , then a Borel set  $B \subset X$  is *full* for  $E$  if  $B$  intersects each  $E$ -class, and that  $F$  is *smooth over  $E$*  if there is a Borel homomorphism  $f: X \rightarrow X$  from  $F$  to  $E$  such that  $x E f(x)$  for all  $x \in X$ .

**5.2. Theorem.** *Let  $E, F$  be countable Borel equivalence relations on the standard Borel spaces  $X, Y$ , respectively, and suppose that  $A$  is an invariant relation, with  $\Phi$  a property of families  $\mathcal{F} \subset A_+$ .*

- (i) *If  $X = Y$  and  $E \subset F$  and  $F$  is Borel non- $(A, \Phi)$ , then  $E$  is Borel non- $(A, \Phi)$ .*
- (ii) *If  $E \leq_B F$  and  $F$  is Borel non- $(A, \Phi)$ , then  $E$  is Borel non- $(A, \Phi)$ .*
- (iii) *If  $B \subset X$  is a Borel subset and  $E$  is Borel non- $(A, \Phi)$  then  $E \restriction B$  is Borel non- $(A, \Phi)$ .*
- (iv) *Suppose that  $B \subset X$  is a Borel subset that is full for  $E$ . If  $E \restriction B$  is Borel non- $(A, \Phi)$ , then  $E$  is Borel non- $(A, \Phi)$ .*
- (v) *Suppose that  $X = Y$ ,  $E \subset F$ , and that  $F$  is smooth over  $E$ . If  $E$  is Borel non- $(A, \Phi)$ , then  $F$  is Borel non- $(A, \Phi)$ .*

*Proof.* Throughout the proofs of (i)–(v) we fix a pairing function  $n \mapsto \langle n_0, n_1 \rangle$  of  $\omega$  onto  $\omega^2$ .

(i) Let  $\phi: X \rightarrow (A_+)^{\omega}$  be a Borel homomorphism from  $E$  to  $E_{\text{set}}(A_+)$ . Suppose  $F$  is the orbit equivalence relation arising from the Borel action of the countable group  $\Gamma = \{\gamma_i : i \in \omega\}$  on  $X$ . For all  $x \in X$  and  $n \in \omega$ , define

$$\tilde{\phi}(x)(n) := \phi(\gamma_{n_0}x)(n_1),$$

so that  $\tilde{\phi}$  is a Borel homomorphism from  $F$  to  $E_{\text{set}}(A_+)$ . Then a Borel homomorphism  $\psi: X \rightarrow A_-$  from  $F$  to  $E_0(A_-)$  will also be a homomorphism from  $E$  to  $E_0(A_-)$ , and

clearly such a homomorphism  $\psi$  will have the property that

$$\forall x \in X \forall n \in \omega \neg(\psi(x) \ A \ \tilde{\phi}(x)(n)) \iff \forall x \in X \forall n \in \omega \neg(\psi(x) \ A \ \phi(x)(n)) .$$

(ii) Suppose that  $E$  is the orbit equivalence relation arising from the Borel action of  $\Gamma = \{\gamma_i : i \in \omega\}$  on  $X$ . Let  $f: X \rightarrow Y$  be a Borel reduction from  $E$  to  $F$ , and let  $\sigma: \text{im}(f) \rightarrow X$  be a Borel function such that  $f \circ \sigma = \text{id}_{\text{im}(f)}$ . Let  $\phi: X \rightarrow (A_+)^{\omega}$  be a Borel homomorphism from  $E$  to  $E_{\text{set}}(A_+)$ , and define the function  $\tilde{\phi}: Y \rightarrow (A_+)^{\omega}$  by

$$\tilde{\phi}(y)(n) := \phi(\gamma_{n_0}\sigma(y))(n_1) ,$$

so that  $\tilde{\phi}$  is a Borel homomorphism from  $F$  to  $E_{\text{set}}(A_+)$ . By hypothesis, obtain a Borel homomorphism  $\tilde{\psi}: Y \rightarrow A_-$  from  $F$  to  $E_0(A_-)$  such that for all  $x \in X$  and  $n \in \omega$ ,

$$\neg(\tilde{\psi}(x) \ A \ \tilde{\phi}(x)(n)) .$$

Let  $\psi = \tilde{\psi} \circ f$ . Then it is easily checked that  $\psi: X \rightarrow A_-$  is a Borel homomorphism from  $E$  to  $E_0(A_-)$  such that for all  $x \in X$  and  $n \in \omega$ ,  $\neg(\psi(x) \ A \ \phi(x)(n))$ .

(iii) Let  $E$  be the orbit equivalence relation arising from the Borel action of  $\Gamma = \{\gamma_i : i \in \omega\}$  on  $X$ , and suppose that  $\phi: B \rightarrow (A_+)^{\omega}$  is a Borel homomorphism from  $E \upharpoonright B$  to  $E_{\text{set}}(A_+)$ . Let  $[B] = \{x \in X : (\exists y \in B) x \ E \ y\}$  be the  $E$ -saturation of  $B$ , and define the function  $\tilde{\phi}: [B] \rightarrow (A_+)^{\omega}$  by

$$\tilde{\phi}(x)(n) := \phi(\gamma_k x)(n), \text{ where } k \text{ is least such that } \gamma_k x \in B .$$

Then  $\tilde{\phi}$  is a Borel homomorphism from  $E \upharpoonright [B]$  to  $E_{\text{set}}(A_+)$ . Extend  $\tilde{\phi}$  arbitrarily to a Borel homomorphism defined on all of  $X$ , and obtain a Borel homomorphism  $\psi: X \rightarrow A_-$  from  $E$  to  $E_0(A_-)$  such that for all  $x \in X$  and for all  $n \in \omega$ ,  $\neg(\psi(x) \ A \ \tilde{\phi}(x)(n))$ . Clearly the restriction of  $\psi$  to  $B$  serves the desired purpose.

(iv) Let  $E$  be the orbit equivalence relation arising from the Borel action of  $\Gamma = \{\gamma_i : i \in \omega\}$  on  $X$ , and suppose that  $\phi: X \rightarrow (A_+)^{\omega}$  is a Borel homomorphism from  $E$  to  $E_{\text{set}}(A_+)$ . Define the function  $\tilde{\phi}: B \rightarrow (A_+)^{\omega}$  by

$$\tilde{\phi}(x)(n) := \phi(\gamma_{n_0}x)(n_1) ,$$

so that  $\tilde{\phi}$  is a Borel homomorphism from  $E \upharpoonright B$  to  $E_{\text{set}}(A_+)$ . Obtain by hypothesis a Borel homomorphism  $\tilde{\psi}: B \rightarrow A_-$  from  $E \upharpoonright B$  to  $E_0(A_-)$  such that for all  $x \in X$  and for all  $n \in \omega$ ,  $\neg(\tilde{\psi}(x) \ A \ \tilde{\phi}(x)(n))$ . Define the function  $\psi: X \rightarrow A_-$  by

$$\psi(x)(n) := \tilde{\psi}(\gamma_k x)(n), \text{ where } k \text{ is least such that } \gamma_k x \in B .$$

Clearly  $\psi$  is a Borel homomorphism from  $E$  to  $E_0(A_-)$  such that for all  $x \in X$  and for all  $n \in \omega$ ,  $\neg(\psi(x) \ A \ \phi(x)(n))$ .

(v) Let  $F$  be the orbit equivalence relation arising from the Borel action of  $\Gamma = \{\gamma_i : i \in \omega\}$  on  $X$ , and let  $f: X \rightarrow X$  be a Borel homomorphism from  $F$  to  $E$  such that  $x E f(x)$  for all  $x \in X$ . Suppose that  $\phi: X \rightarrow (A_+)^{\omega}$  is a Borel homomorphism from  $F$  to  $E_{\text{set}}(A_+)$ , and notice that  $\phi$  is also a homomorphism from  $E$  to  $E_{\text{set}}(A_+)$ . By hypothesis, obtain a Borel homomorphism  $\tilde{\psi}: X \rightarrow A_-$  from  $E$  to  $E_0(A_-)$  such that for all  $x \in X$  and for all  $n \in \omega$ ,  $\neg(\tilde{\psi}(x) A \phi(x)(n))$ . Now define the function  $\psi: X \rightarrow A_-$  by

$$\psi(x)(n) := \tilde{\psi}(\gamma_k x)(n), \text{ where } k \text{ is least such that } \gamma_k x E f(x).$$

It is easily checked that  $\psi$  is a Borel homomorphism from  $F$  to  $E_0(A_-)$  such that for all  $x \in X$  and for all  $n \in \omega$ ,  $\neg(\psi(x) A \phi(x)(n))$ .  $\square$

Next we discuss the relationships between Borel invariant properties, which frequently correspond to inequalities between cardinal invariants. Many of the inequalities expressed in van Douwen's diagram are captured by combinatorial gadgets called *generalized Galois-Tukey connections*, or *morphisms* for short. If  $A$  and  $B$  are relations, then a morphism from  $A$  to  $B$  is a pair of functions

$$\xi_- : B_- \rightarrow A_-$$

$$\xi_+ : A_+ \rightarrow B_+$$

such that for all  $b \in B_-$  and  $a \in A_+$ ,

$$\xi_-(b) A a \implies b B \xi_+(a).$$

The definition is precision-engineered to yield the following.

**5.3. Proposition.** *If there exists a morphism from  $A$  to  $B$ , then  $\|A\| \geq \|B\|$ .*

*Proof.* We shall show that if  $\mathcal{F} \in D(A)$ , then  $\xi_+(\mathcal{F}) \in D(B)$ , which suffices since the cardinality of  $\xi_+(\mathcal{F})$  is less than or equal to that of  $\mathcal{F}$ . Indeed,

$$\begin{aligned} (\forall x \in A_-)(\exists y \in \mathcal{F})(x A y) &\implies (\forall \xi_-(z) \in A_-)(\exists y \in \mathcal{F})(\xi_-(z) A y) \\ &\implies (\forall \xi_-(z) \in A_-)(\exists y \in \mathcal{F})(z B \xi_+(y)) \\ &\implies (\forall z \in B_-)(\exists w \in \xi_+(\mathcal{F}))(z B w), \end{aligned}$$

as desired.  $\square$

There is an obvious generalization to arbitrary tame cardinal invariants: if there exists a morphism  $(\xi_-, \xi_+)$  from  $A$  to  $B$  such that whenever  $\mathcal{F}$  has property  $\Phi$  then  $\xi_+(\mathcal{F})$  has property  $\Psi$ , then  $\|A\|_{\Phi} \geq \|B\|_{\Psi}$ .

5.4. *Example.* To show that  $\mathfrak{p} \leq \mathfrak{a}$ , one may simply observe that whenever the centered family  $\{a_\alpha\}$  has no pseudo-intersection, then  $\{a_\alpha^c\}$  is maximal and almost disjoint. To prove this with morphisms, let  $\xi_-(b) = b$  and  $\xi_+(a) = a^c$ ; then  $\xi_+$  takes infinite almost disjoint families to families with the strong finite intersection property, and whenever  $b \subset^* a^c$ , we have that  $b$  is almost disjoint from  $a$ .

Although the existence of a morphism from  $A$  to  $B$  implies that  $\|A\| \geq \|B\|$ , in the study of cardinal invariants one is ultimately interested in determining *which models* of set theory satisfy a given cardinal inequality. The most important relationships between cardinal invariants are the true inequalities, that is, those which are provable in ZFC. In [Bla96], Blass notes that even if there is morphism from  $A$  to  $B$ , the inequality  $\|A\| \geq \|B\|$  need not be provable in ZFC. In that paper Blass proposes that one look instead for *definable* morphisms, and he establishes the following.

5.5. **Theorem** (Blass). *Suppose that there is a Borel morphism from  $A$  to  $B$ , meaning that the components  $\xi_-$ ,  $\xi_+$  are Borel functions. Then the inequality  $\|A\| \geq \|B\|$  cannot be violated by forcing.*

For our purposes, even Borel morphisms are insufficient since the existence of such a morphism from  $A$  to  $B$  does not yield a proof that Borel non- $B$  implies Borel non- $A$ . Moreover, it is sometimes the case that Borel non- $B$  implies Borel non- $A$ , and yet there is no morphism from  $A$  to  $B$  at all. For instance, we shall show in the next section that  $\mathfrak{s} \rightarrow \mathfrak{r}$ , but of course there are models in which  $\mathfrak{s} < \mathfrak{r}$ . What we need instead is the following.

5.6. **Definition.** Let  $A, B$  be invariant relations. A Borel morphism  $(\xi_-, \xi_+)$  from  $A$  to  $B$  is said to be *invariant* iff  $\xi_-$  is a Borel homomorphism from  $E_0(B_-)$  to  $E_0(A_-)$ .

5.7. **Theorem.** *If there exists an invariant Borel morphism from  $A$  to  $B$ , then the property Borel non- $B$  implies Borel non- $A$ .*

By our earlier remarks, it is still possible for there to exist a Borel morphism between two relations but no invariant Borel morphism.

*Proof.* Let  $(\xi_-, \xi_+)$  be the invariant Borel morphism from  $A$  to  $B$ . Let  $E$  be a countable Borel equivalence relation with the property inspired by  $B$ . Given a Borel homomorphism  $\phi: E \rightarrow E_{\text{set}}(A_+)$ , we shall consider the homomorphism  $\phi': E \rightarrow E_{\text{set}}(B_+)$  defined by

$$\phi'(x)(n) := \xi_+(\phi(x)(n)) .$$

Since  $E$  is Borel non- $B$ , there exists a homomorphism  $\psi': E \rightarrow E_0(B_-)$  such that for all  $x \in X$  and  $n \in \omega$  we have  $\neg(\psi'(x) B \phi'(x)(n))$ . Letting  $\psi = \xi_- \circ \psi'$ , we have that

$\psi: E \rightarrow E_0(A_-)$  is a Borel homomorphism and for all  $x \in X, n \in \omega$ ,

$$\begin{aligned} \neg(\psi'(x) B \phi'(x)(n)) &\implies \neg(\psi'(x) B \xi_+(\phi(x)(n))) \\ &\implies \neg(\xi_-(\psi'(x)) A \phi(x)(n)) \\ &\implies \neg(\psi(x) A \phi(x)(n)) . \end{aligned}$$

Hence  $E$  is Borel non- $A$ . □

As in the case of ordinary morphisms, it is possible to generalize this to arbitrary simple Borel invariant properties. To show that property Borel non- $(B, \Psi)$  implies Borel non- $(A, \Phi)$ , it suffices to find an invariant Borel morphism  $(\xi_-, \xi_+)$  from  $A$  to  $B$  such that  $\xi_+(\mathcal{F})$  has property  $\Psi$  whenever  $\mathcal{F}$  has property  $\Phi$ .

We now conclude this section with a generalization of a result of Thomas which shows that under the hypothesis of Martin's Conjecture, there exists a countable Borel equivalence relation which is *not* Borel bounded. Recall that  $\leq_T$  denotes the Turing reducibility relation on  $2^\omega$ . Then the *Turing equivalence relation*  $\equiv_T$  defined by  $x \equiv_T y$  iff  $x \leq_T y$  and  $y \leq_T x$  is one of the most important countable Borel equivalence relations.

Martin's Conjecture is the statement that any Borel  $\equiv_T$ -invariant function is either constant or increasing on a cone. (Here, a subset of  $2^\omega$  is said to be a *cone* if it is  $\leq_T$ -upwards closed). We shall require only the following consequence of Martin's Conjecture (for instance, see Theorem 2.1(i) in [Tho09]):

**5.8. Lemma.** *Assuming Martin's conjecture, if  $f: 2^\omega \rightarrow 2^\omega$  is a Borel homomorphism from  $\equiv_T$  to  $E_0$  then there exists a cone  $C$  such that  $f(C)$  is contained in a single  $E_0$ -class.*

Using this, Thomas proved in Theorem 5.2 of [Tho09] that Martin's Conjecture implies that  $\equiv_T$  is not Borel bounded. We now show that his argument applies to *any* (nontrivial) simple Borel invariant property.

**5.9. Theorem.** *Let  $A$  be an invariant relation, and assume that for all  $z \in A_-$  there exists  $a \in A_+$  such that  $z A a$ . Assuming Martin's Conjecture, the Turing equivalence relation  $\equiv_T$  is not Borel non- $A$ .*

*Proof.* Suppose towards a contradiction that  $\equiv_T$  is Borel non- $A$ . Let  $\phi: 2^\omega \rightarrow A_+$  be any Borel function such that for all  $a \in A_+$ , the preimage  $\phi^{-1}(a)$  is cofinal. (To see that there exists such a map, let  $x \mapsto \langle x_0, x_1 \rangle$  be a pairing function on  $2^\omega$  and let  $f$  be a Borel bijection between  $2^\omega$  and  $A_+$ . Then  $\phi(x) := f(x_0)$  has the desired properties).

It follows that for all  $a \in A_+$ , the saturation  $[\phi^{-1}(a)]_{\equiv_T}$  contains a cone. Since  $\equiv_T$  is Borel non- $A$ , there exists a homomorphism  $\psi: E \rightarrow E_0(A_-)$  such that for all  $x \in 2^\omega$ ,

$$\neg(\psi(x) A \phi(x))$$

By Lemma 5.8, there exists a cone  $C$  such that  $\psi(C)$  is contained in a single  $E_0(A_-)$ -class, say  $[z]_{E_0}$ . Since  $A$  is nontrivial, there exists  $a \in A_+$  such that  $z A a$ . Since  $[\phi^{-1}(a)]_{\equiv_T}$  contains a cone, it meets  $C$ . In particular, there exists  $x \in 2^\omega$  such that  $\psi(x) A \phi(x)$ , which is a contradiction.  $\square$

## 6. THE NON-SPLITTING PROPERTY

The non-splitting property  $s$  would appear to be very special, as it does not hold of any non-trivial equivalence relations.

**6.1. Theorem.** *If the countable Borel equivalence relation  $E$  has property  $s$ , then  $E$  is smooth.*

For the proof we shall require the following standard measure-theoretic fact. Let  $m$  denote the Haar (or “coin-tossing”) measure on  $2^\omega$ , so that  $m$  is the  $\omega$ -fold product of the  $(\frac{1}{2}, \frac{1}{2})$  measure on  $\{0, 1\}$ . Then  $[\omega]^\omega$  is an  $E_0$ -invariant, Borel,  $m$ -conull subset of  $2^\omega$ , and we denote the restriction of  $m$  to  $[\omega]^\omega$  also by  $m$ .

**6.2. Proposition.** *Suppose that  $N \subset [\omega]^\omega$  has the following properties:*

- (a)  $N$  is  $E_0$ -invariant
- (b) *for any  $x \in [\omega]^\omega$ , exactly one of  $x$  or  $x^c$  is in  $N$*

*Then  $N$  is not  $m$ -measurable.*

*Proof.* Suppose that  $N$  is measurable. We consider the natural bitwise action of  $G := \bigoplus_{i \in \omega} \mathbb{Z}/2\mathbb{Z}$  on  $[\omega]^\omega$ . Observe that a subset of  $[\omega]^\omega$  is  $E_0$ -invariant iff it is  $G$ -invariant. It is well-known that this action of  $G$  is *ergodic*, meaning that every  $G$ -invariant measurable subset of  $[\omega]^\omega$  has measure 0 or 1. Hence, property (a) implies that  $N$  has measure 0 or 1. On the other hand, property (b) implies that  $N$  has measure  $\frac{1}{2}$ , since the map  $x \mapsto x^c$  is a measure-preserving bijection which sends  $N$  onto  $[\omega]^\omega \setminus N$ .  $\square$

*Proof of Theorem 6.1.* By Theorem 5.2, it suffices to show that there exists a hyperfinite equivalence relation  $E$  which does not have property  $s$ . To this end, let  $E$  be the hyperfinite Borel equivalence relation on  $X = [\omega]^\omega$  given by

$$x E x' \iff x E_0 x' \text{ or } x^c E_0 x',$$



and assume that  $E$  is the orbit equivalence relation arising from the Borel action of the countable group  $\Gamma = \{\gamma_i : i \in \omega\}$  on  $X$ .

Now suppose that  $E$  is Borel non-splitting, and define the Borel homomorphism  $\phi: X \rightarrow X^\omega$  from  $E$  to  $E_{\text{set}}$  by

$$\phi(x)(n) = \gamma_n x ,$$

so that each  $\phi(x)$  enumerates the family  $[x]_E$ . Let  $\psi: X \rightarrow X$  be a Borel homomorphism from  $E$  to  $E_0$  such that for all  $x$  and for all  $n$ ,

$$\text{either } \psi(x) \subset^* \phi(x)(n) \text{ or } \psi(x) \subset^* \phi(x)(n) .$$

Then in particular, for each  $x$  we have either  $\psi(x) \subset^* x$  or  $\psi(x) \subset^* x^c$ . Now put

$$N := \{x \in X : \psi(x) \subset^* x\} .$$

Let us check that  $N$  satisfies conditions (a) and (b) of Proposition 6.2. For (a), suppose that  $x \in N$ , so that  $\psi(x) \subset^* x$ , and let  $x'$  be such that  $x' E_0 x$ . Then we have  $\psi(x') E_0 \psi(x)$ , so that  $\psi(x') =^* \psi(x) \subset^* x =^* x'$ , and hence  $x' \in N$  too. For condition (b), suppose that  $x \in N$  so that  $\psi(x) \subset^* x$ . Then since  $x^c E x$ , we have  $\psi(x^c) E_0 \psi(x)$ , so that  $\psi(x^c) =^* \psi(x) \subset^* x$ . It follows that  $\psi(x^c) \not\subset^* x^c$ , and hence  $x^c \notin N$ . This shows that  $x \in N$  implies  $x^c \in 2^\omega \setminus N$ , and the converse is the same.

It now follows from Proposition 6.2 that  $N$  is not  $m$ -measurable. But from the definition of  $N$  it is clear that it is Borel, and this is a contradiction.  $\square$

As a corollary, we also obtain a characterization of the following Borel invariant properties which are inspired by the cardinal invariants  $\text{par}_n$ .

**6.3. Definition.** Let  $E$  be a countable Borel equivalence relation on the standard Borel space  $X$ . Then  $E$  has property  $\text{par}_n$  if for every Borel homomorphism  $\phi: X \rightarrow (2^{\omega^n})^\omega$  from  $E$  to  $E_{\text{set}}(2^{\omega^n})$ , there exists a Borel homomorphism  $\psi: X \rightarrow 2^{\omega^n}$  from  $E$  to  $E_0$  such that for all  $x \in X$ ,  $\psi(x)$  is infinite and almost homogeneous for each function  $\phi(x)(n)$ . Here,  $A \subset \omega$  is *almost homogeneous* for a function  $f$  if  $A$  is almost equal to a set which is homogeneous for  $f$ .

**6.4. Corollary.** *If  $E$  has property  $\text{par}_n$ , then  $E$  is smooth.*

*Proof.* This result follows from Theorem 6.1 using the following two easy observations. First, property  $\text{par}_1$  coincides with property  $s$ ; and second,  $\text{par}_n \rightarrow \text{par}_m$  for  $n \geq m$ .  $\square$

We close this section by noting that the converse of Theorem 6.1 also holds. Indeed, each of the properties in Definition 4.6 holds of smooth relations. For properties other than

s, this is implicit in Proposition 7.2 together with Theorems 7.1 and 5.2, but intuitively, if  $E$  is a smooth countable Borel equivalence relation on the standard Borel space  $X$  with Borel transversal  $B \subset X$ , then for a given property  $\times$  corresponding to the cardinal  $\mathfrak{x}$ , we simply use the unique point in  $[x]_E \cap B$  as the base point for the appropriate diagonalization argument against  $\{\phi(x)(n) : n \in \omega\}$  which shows that  $\aleph_0 < \mathfrak{x}$ .

This is perhaps hardest to see in the case of  $\mathfrak{s}$ , since proving  $\aleph_0 < \mathfrak{s}$  is slightly more complicated than proving the corresponding fact for the other cardinal invariants we consider. For completeness we quickly sketch the argument here.

Thus suppose  $E$  is a smooth countable Borel equivalence relation on the standard Borel space  $X$ , with Borel transversal  $B \subset X$  and Borel function  $\sigma: X \rightarrow X$  such that for each  $x \in X$ ,  $\sigma(x)$  is the unique element in  $B \cap [x]_E$ . Suppose  $\phi: X \rightarrow ([\omega]^\omega)^\omega$  is an arbitrary Borel homomorphism from  $E$  into  $E_{\text{set}}$ . Inductively define the Borel functions  $\phi'_n: X \rightarrow [\omega]^\omega$  by  $\phi'_0(x) = \phi(\sigma(x))(0)$  and

$$\phi'_{n+1}(x) = \begin{cases} (\phi \circ \sigma)(x)(n+1) \cap \phi'(x)(n) & \text{if this set is infinite} \\ (\phi \circ \sigma)(x)(n+1)^c \cap \phi'(x)(n) & \text{otherwise.} \end{cases}$$

Then inductively define the Borel functions  $\alpha_n: X \rightarrow \omega$  by

$$\alpha_n(x) = \min [\phi'_n(x) \setminus \{\alpha_k(x) : k < n\}] ,$$

and finally set  $\psi(x)(n) = \alpha_n(x)$ . It is easily checked that  $\psi(x) \subset^* \phi(x)(n)$  or  $\psi(x) \subset^* \phi(x)(n)^c$  for each  $x \in X$  and  $n \in \omega$ . Moreover,  $\psi$  is not merely a Borel homomorphism from  $E$  to  $=^*$ , but in fact is  $E$ -invariant.

## 7. PROOF OF THE DIAGRAM

In this section, we shall prove the implications among Borel invariant properties which are indicated in Figure 4. We also briefly discuss possible reasons why some of the arrows are “missing” when compared with Figure 3.

**7.1. Theorem.** *The arrows in Figure 4 correspond to true implications between Borel invariant properties.*

The first implication,  $\mathfrak{s} \rightarrow \mathfrak{p}$ , follows somewhat vacuously from Theorem 6.1 together with the easy fact that smooth equivalence relations have property  $\mathfrak{p}$ . In fact, this implication is not reversible, since there also exist nonsmooth relations with property  $\mathfrak{p}$ .

**7.2. Proposition.** *If  $E$  is hyperfinite, then  $E$  has property  $\mathfrak{p}$ .*

*Proof.* Express  $E$  as the increasing union of finite Borel equivalence relations  $F_n$ , and suppose we are given a Borel homomorphism  $\phi: X \rightarrow ([\omega]^\omega)^\omega$  from  $E$  to  $E_{\text{set}}$  such that for each  $x \in X$ , the family  $\{\phi(x)(n) : n \in \omega\}$  is centered. We inductively define functions  $a_n: X \rightarrow \omega$  as follows. Given  $a_0(x), \dots, a_n(x)$ , let

$$a_{n+1}(x) := \min \left[ \left( \bigcap_{i \leq n} \phi(x)(i) \right) \setminus \{a_i(y) : i \leq n \text{ and } y F_n x\} \right].$$

This construction is just a slight reorganizing of the usual diagonalization, so if we let  $\psi(x) := \{a_n(x) : n \in \omega\}$ , then it is easy to see that  $\psi(x)$  is a pseudo-intersection of the set  $\{\phi(x)(n) : n \in \omega\}$ . Moreover, the  $a_n$  have the property that if  $x E y$ , then the sequences  $\langle a_n(x) : n \in \omega \rangle$  and  $\langle a_n(y) : n \in \omega \rangle$  will eventually be equal, so that  $\psi$  is also a homomorphism from  $E$  to  $E_0$ .  $\square$

Currently,  $s \rightarrow p$  and its consequences are the only implications that we can prove are nonreversible.

The remainder of the proof of Theorem 7.1 will be given in a series of lemmas. In most cases the proofs amount to trivial observations, such as noticing that the standard proof of the corresponding cardinal inequality can be carried out in a Borel and invariant fashion. However, in a few cases some care is needed to make sure that this can be done.

**7.3. Lemma.**  $p \rightarrow t$ .

*Proof.* This holds simply because every tower is a centered family.  $\square$

**7.4. Lemma.**  $p \rightarrow a$ .

*Proof.* It suffices to observe that the morphism  $\xi_-(b) = b$ ,  $\xi_+(a) = a^c$  described in Example 5.4 is Borel and invariant.  $\square$

**7.5. Lemma.**  $p \rightarrow b$ .

This is a difficult case in which the classical proof that  $p \leq b$  apparently cannot be carried out in an invariant fashion. We were able to obtain only the weaker result  $p \rightarrow u$ , and the problem remained open until Tamás Mátrai and Juris Steprāns provided us with a positive answer. Since the proof will appear elsewhere, we give only a brief outline here.

To establish  $p \rightarrow b$ , one requires a morphism  $\xi_-: [\omega]^\omega \rightarrow \omega^\omega$ ,  $\xi_+: \omega^\omega \rightarrow [\omega]^\omega$  such that  $\text{im}(\xi_+)$  is centered and

$$A \subset^* \xi_+(f) \implies f \leq^* \xi_-(A).$$

To do this one constructs the map  $\xi_+$  with the additional property that for every  $\leq^*$ -unbounded subset  $S \subset \omega^\omega$ , the set  $\xi_+(S)$  does not have a pseudo-intersection. It follows that for each  $A \in [\omega]^\omega$ , the set  $S_A := \{f \in \omega^\omega : A \subset^* \xi_+(f)\}$  is  $\leq^*$ -bounded. Letting  $\xi_-(A)$  be such a bound, it is easy to see that  $\xi_-$ ,  $\xi_+$  satisfy the required properties. Finally, it is possible to compute the bounds  $\xi_-(A)$  in a Borel fashion.<sup>2</sup>

**7.6. Lemma.**  $\mathfrak{b} \rightarrow \mathfrak{d}$ .

*Proof.* This holds trivially because any function which bounds a family also witnesses that the family is not dominating.  $\square$

**7.7. Lemma.**  $\mathfrak{b} \rightarrow \mathfrak{r}$ .

*Proof.* We follow the proof that  $\mathfrak{b} \leq \mathfrak{r}$  which is given in [Bla03]. Given a subset  $x \subset \omega$ , we let  $f_x: \omega \rightarrow \omega$  be a function such that each interval  $[n, f_x(n))$  contains an element of  $x$ . Now, given a homomorphism  $\phi: X \rightarrow ([\omega]^\omega)^\omega$  from  $E$  to  $E_{\text{set}}$ , we apply property  $\mathfrak{b}$  to the function  $x \mapsto f_{\phi(x)}$  to obtain a homomorphism  $\psi: X \rightarrow [\omega]^\omega$  from  $E$  to  $=^*$  such that for all  $x$ ,  $\psi(x)$  eventually dominates  $f_{\phi(x)}$ .

*Claim.* There exists a Borel homomorphism  $\lambda: \omega^\omega \rightarrow \omega^\omega$  from  $=^*$  to  $=^*$  such that for every  $f \in \omega^\omega$ , every interval  $[\lambda(f)(j), \lambda(f)(j+1))$  contains an interval of the form  $[n, f(n))$ .

*Proof of claim.* Begin by expressing  $=^*$  as an increasing union of finite Borel equivalence relations  $F_i$ . We inductively define an increasing sequence of functions  $j_i: \omega^\omega \rightarrow \omega$  as follows. Given  $j_i$ , define an auxiliary function  $k_{i+1}$  so that for all  $f$ , the interval  $[j_i(f), k_{i+1}(f))$  contains an interval of the form  $[n, f(n))$ . Then, let

$$j_{i+1}(f) := \max \{k_{i+1}(g) : g F_i f\}$$

We now let  $\lambda(f) := \langle j_i(f) \rangle$ , and it is clear that  $\lambda$  is as desired.  $\dashv$

We now consider the composition  $\lambda \circ \psi$ , which has the property that for all  $m$ , almost every  $[\lambda \circ \psi(x)(n), \lambda \circ \psi(x)(n+1))$  contains an element of  $\phi(x)(m)$ . We may therefore define

$$\psi'(x) := \bigcup_{n \text{ odd}} [(\lambda \circ \psi)(x)(n), (\lambda \circ \psi)(x)(n+1))$$

and we will have that  $\psi'(x)$  splits each member of the family enumerated by  $\phi(x)$ .  $\square$

**7.8. Lemma.**  $\mathfrak{r} \rightarrow \mathfrak{u}$

<sup>2</sup>In this construction,  $\xi_-$  will be  $E_0$ -invariant. In fact we can obtain this extra property automatically by the following general observation: If there is a Borel morphism from  $A$  to  $B$ ,  $\neg A$  is transitive, and  $E_0$  is Borel non- $A$ , then there is an *invariant* Borel morphism from  $A$  to  $B$ .

*Proof.* This holds just because if some set  $Y$  witnesses that a family  $\mathcal{F}$  is not unsplittable, then it also witnesses that  $\mathcal{F}$  cannot be an ultrafilter base.  $\square$

**7.9. Lemma.**  $r \rightarrow i$ .

*Proof.* Suppose the homomorphism  $\phi$  from  $E$  to  $E_{\text{set}}$  has the property that every  $\phi(x)$  enumerates an independent family. Let  $\phi'(x)$  enumerate all possible intersections of finitely many sets from  $\phi(x)$  with the complements of finitely many other sets from  $\phi(x)$ . Since  $E$  has property  $r$ , there exists a homomorphism  $\psi$  from  $E$  to  $E_0$  such that for every  $x$  and  $n \in \omega$ ,  $\psi(x)$  splits  $\phi'(x)(n)$ . Then  $\psi(x)$  also witnesses that the family  $\phi(x)$  is not maximal independent.  $\square$

This completes the proof of Theorem 7.1.

We close with a brief discussion of the remaining implications, which we were not able to prove. With the exception of implications involving  $s$ , it appears to be very difficult to establish non-implications between the invariant properties. However, in many cases we can at least show that our primary method of establishing the implications fails by proving that there does not exist a Borel invariant morphism witnessing the implication in question. In other words, when we cannot determine the relationship between invariant Borel properties, we content ourselves with determining their relationship in the Borel Tukey ordering (or better: in the invariant Borel Tukey ordering).

For instance, since it is well-known that van Douwen's diagram is forcing complete, Theorem 5.5 implies that all of the inequalities *not* present in van Douwen's diagram cannot be witnessed by Borel morphisms. (Here we mean to except  $t \leq p$ , which is not known.) Additionally, we have:

**7.10. Theorem.** *The true inequality  $\mathfrak{b} \leq \mathfrak{a}$  is not witnessed by a Borel morphism.*

*Proof.* Suppose that there is a Borel pair of maps  $\xi_-: \omega^\omega \rightarrow [\omega]^\omega$  and  $\xi_+: [\omega]^\omega \rightarrow \omega^\omega$  satisfying:

$$(7.11) \quad \xi_+(A) \leq^* f \implies |A \cap \xi_-(f)| < \aleph_0.$$

(To avoid trivialities, we only suppose this implication holds for co-infinite  $A$ .) Since  $\xi_-, \xi_+$  are Borel, this implication still holds after forcing to add a random real  $\dot{A}$ . Now, it is well-known that random forcing is  $\omega^\omega$ -bounding, which implies that there exists an element  $f \in \omega^\omega$  of the ground model such that  $\xi_+(\dot{A}) \leq^* f$ . But it is easy to see that a random real is always a splitting real, and hence  $|\dot{A} \cap \xi_-(f)| = \aleph_0$ , contradicting Equation 7.11.  $\square$

It would be very useful to have a more complete picture of the Borel Tukey ordering between the combinatorial cardinal invariants. For instance, we do not know the status of the remaining inequalities of interest to us:  $\mathfrak{d} \leq \mathfrak{i}$  and  $\mathfrak{t} \leq \mathfrak{b}$ . It is tempting to conjecture that they too cannot be witnessed by Borel invariant morphisms. But considering the subtlety of the morphism constructed for Lemma 7.5, it is not clear whether this is the case.

## REFERENCES

- [BJ07] Charles M. Boykin and Steve Jackson. Borel boundedness and the lattice rounding property. In *Advances in logic*, volume 425 of *Contemp. Math.*, pages 113–126. Amer. Math. Soc., Providence, RI, 2007.
- [Bla96] Andreas Blass. Reductions between cardinal characteristics of the continuum. In *Set theory (Boise, ID, 1992–1994)*, volume 192 of *Contemp. Math.*, pages 31–49. Amer. Math. Soc., Providence, RI, 1996.
- [Bla03] Andreas Blass. Combinatorial cardinal invariants. In Matthew Foreman and Akihiro Kanamori, editors, *The handbook of set theory*. Springer, 2003.
- [DJK94] R. Dougherty, S. Jackson, and A. S. Kechris. The structure of hyperfinite Borel equivalence relations. *Trans. Amer. Math. Soc.*, 341(1):193–225, 1994.
- [JKL02] S. Jackson, A. S. Kechris, and A. Louveau. Countable Borel equivalence relations. *J. Math. Log.*, 2(1):1–80, 2002.
- [Kec95] Alexander S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.
- [Tho09] Simon Thomas. Martin’s conjecture and strong ergodicity. *Arch. Math. Logic*, 48(8):749–759, 2009.
- [Voj93] P Vojtáš. Generalized galois-tukey connections between explicit relations on classical objects of real analysis. In H. Judah, editor, *Set Theory of the Reals*, volume 6 of *Israel Mathematical Conference Proceedings*, pages 619–643. Amer. Math. Soc., 1993.
- [Zap04] Jindřich Zapletal. Descriptive set theory and definable forcing. *Mem. Amer. Math. Soc.*, 167(793):viii+141, 2004.

SAMUEL COSKEY ◊ THE FIELDS INSTITUTE ◊ 222 COLLEGE STREET ◊ TORONTO, ON M5T 3J1 ◊ CANADA  
*E-mail address:* `scoskey@nylogic.org`

SCOTT SCHNEIDER ◊ MATHEMATICS AND COMPUTER SCIENCE DEPARTMENT, WESLEYAN UNIVERSITY ◊  
 SCIENCE TOWER 655 ◊ 265 CHURCH STREET ◊ MIDDLETOWN, CT 06459  
*E-mail address:* `sschneider01@wesleyan.edu`